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Birkhoff Signature Change: A Criterion for the Instability of Chaotic Resonance

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Birkhoff signature change: a criterion for the instability of chaotic resonance

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For periodically forced nonlinear oscillators permitting escape from a potential well a relation is observed between two well-known phenomena, the period-doubling cascade leading to the chaotic escape of the resonant attractor and the complex dynamics associated with the creation of a structurally unstable homoclinic orbit. The particular homoclinic orbit is identified as that created at the initial change of the period one Birkhoff signature of the invariant manifolds of the hilltop saddle. The primary resonant attractor may thus be viewed as the period one simple Newhouse orbit. Significant subharmonic and superharmonic escape events may likewise be associated with nearby Birkhoff signature changes. Significant information about the global dynamics may thus be obtained with little numerical effort by inspection of the signatures of the invariant manifolds of the hilltop saddle.

1. Introduction

The behaviour of many engineering dynamical systems can be formulated in terms of motion in a potential well and of particular significance are those systems where failure corresponds to escape from the well. For example, a dynamical system may be represented by an equation of the form

$$\ddot{x}+c(x,\dot{x})+g(x)=F(t),$$

where x is the displacement, $c(x,\dot{x})$ a damping term, dissipative such that $\operatorname{sgn}[c(x,\dot{x})]=\operatorname{sgn}[\dot{x}]$, $g(x)$ is the nonlinear displacement-dependent restoring force and $F(t)$ is an applied periodic excitation with zero mean. This has a corresponding potential function $\mathscr{V}(x)$ of slope $\mathrm{d}\mathscr{V}/\mathrm{d}x=g(x)$. For an important class of systems the potential function has a form broadly described by figure 1 such that under certain forcing conditions the system may escape from the well to arbitrarily large displacement beyond the potential maximum. For small amplitudes of dynamic forcing the behaviour of a physical system starting from the static equilibrium at the bottom of the well may be adequately modelled by linearity assumptions supplemented with perturbation techniques. Under increasing forcing the amplitude of the motions can grow such that the system explores regions of the well where the assumptions of linearity are less applicable and familiar nonlinear phenomena such as coexistent solutions, jumps to resonance and period-doubling cascades may be encountered. Some typical frequency curves under increasing forcing are provided in figure 2*a*–*d* (see also McRobie & Thompson (1990), for example). One typical feature is a zone of resonant hysteresis (figure 2*b*), where stable resonant and non-resonant period one motions are separated by a directly unstable saddle and the zone is bounded by saddle-node bifurcation arcs A and B where the intervening saddle

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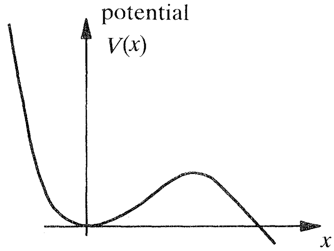

Figure 1. Archetypal potential well for a simple nonlinear engineering oscillator permitting escape over a potential maximum under periodic forcing.

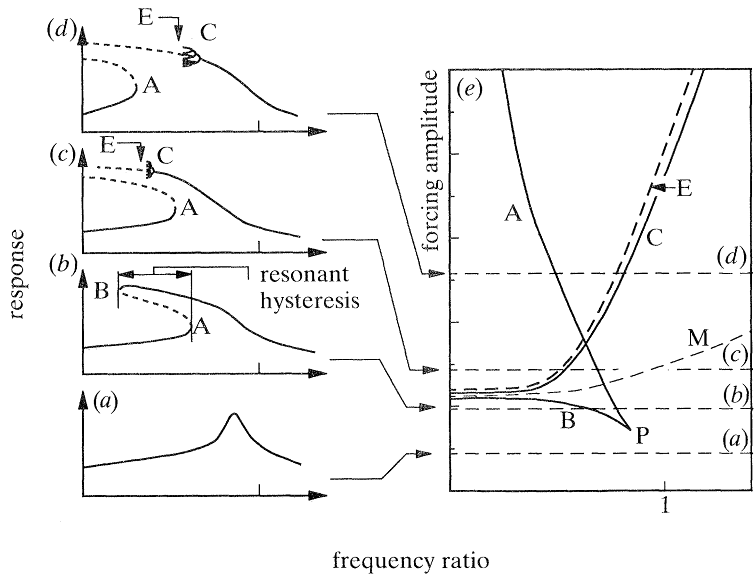

Figure 2. (*a*)–(*d*) Typical frequency–response curves under increasing forcing amplitudes for an oscillator having a potential well of the form shown in figure 1. (*e*) is the corresponding bifurcation diagram at constant damping. Arcs A and B correspond to saddle-node bifurcations, C is the first period-doubling (or flip) bifurcation of the resonant attractor, and E is the chaotic escape event where the resulting chaotic attractor loses stability. Arc M corresponds to the initial homoclinic tangency of the invariant manifolds of the hilltop saddle cycle. Subharmonic and superharmonic activity is omitted for clarity.

collides with one of the attractors. At higher forcing values (figure 2*c*) the resonant attractor can undergo period-doubling cascades (starting on arc C) leading to a numerically observed chaotic attractor which can be subsequently destroyed at a chaotic escape event (arc E). This is a global bifurcation involving collision of the attractor with the global basin boundary.

Knowledge of the locus of the bifurcations in parameter space (figure 2*e*) provides information about how a physical system will evolve as parameters are varied. Of particular significance is the chaotic escape arc E, the locus of those parameter values where the resonant attractor, now chaotic, is destroyed, since any system following that attractor path is then liable to escape.

2. Preliminaries

Analysis of simple periodically forced nonlinear oscillators is most suitably approached using the method *Poincaré sections* (see, for example, Thompson & Stewart 1986). The velocity and displacement of a system are stroboscopically sampled at successive periods of the applied forcing. Under one cycle of the applied forcing, the study of the three-dimensional flow reduces to analysis of the action of a two-dimensional diffeomorphism mapping the Poincaré plane $(x, \dot{x})$ onto itself.

For simple-sided escape systems such as typified by the potential of figure 1, there generally exists over wide parameter ranges an unstable periodic solution referred to as the *hilltop saddle cycle* which oscillates about the local potential maximum. It is demonstrated in McRobie & Thompson (1992*a*), for example, how the invariant manifolds of this solution are closely related to the location of attractors, saddles, basins of attraction, and in general all invariant sets of the diffeomorphism.

A *homoclinic orbit* is one that is asymptotic in both forward and reverse time to some given periodic orbit. It is well known that the existence of such homoclinic orbits 'implies chaos', in that the Smale–Birkhoff homoclinic theorem (see, for example, Guckenheimer & Holmes 1983) implies the existence of horseshoes in the vicinity of any homoclinic orbit. In the analysis of simple nonlinear oscillators the proof of 'chaos' is often simply established by use of Melnikov's method to prove the existence of homoclinic orbits. However, such approaches often have little relevance to the dominant features of the dynamics (such as the chaotic escape of the resonant attractor). The intention of this paper is to highlight an observed relation between one specific orbit (which is homoclinic to the hilltop saddle) and the primary chaotic escape event.

3. Chaotic escape

The chaotic escape of the primary resonant attractor in simple periodically forced nonlinear oscillators is an event variously referred to as a blue-sky catastrophe (see, for example, Abraham & Stewart 1986), a chaotic saddle catastrophe (Stewart 1987) or a boundary crisis (Grebogi *et al.* 1983). Figure 3 is a typical Poincaré section taken from a system near such a crisis, showing a chaotic attractor close to touching a fractal basin boundary. Historically a great deal of theoretical and numerical attention has been focused on these phenomena (see, for example, Stewart & Ueda 1991; Ueda *et al.* 1990; Kleczka *et al.* 1989), such that the sequence of events observed on passage through such a crisis can be described in great detail. For example, in Thompson (1989) it is described how the period-doubling cascade emanating from the resonant attractor leads to a chaotic attractor which subsequently collides with a periodic, directly unstable saddle (referred to as the *destroyer*) which is contained in the global basin boundary. Coincident with this loss of stability there are tangencies in the invariant manifolds of the destroyer and other saddles in the chaotic attractor.

Despite the availability of detailed descriptions of the phenomenon, various awkward factors remain.

1. Most of the important results have yet to be rigorously established and many features of such descriptions are at the level of numerical observation, rather than mathematical surety. For example, it does not yet appear to be proven whether or not the observed 'chaotic' attractor is indeed truly 'chaotic' (although Benedicks & Carlesson (1991) have certain related results in this direction).

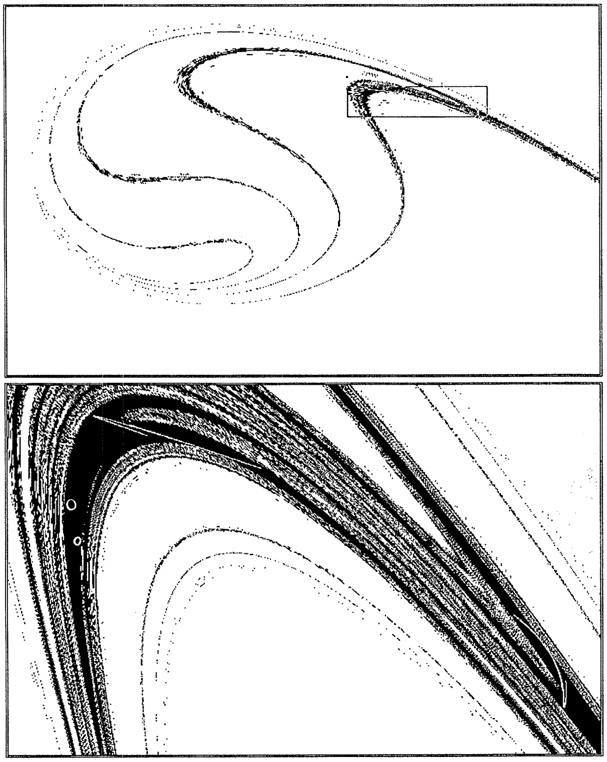

Figure 3. Cell maps of a chaotic attractor close to a boundary crisis. The system is the escape equation of Thompson (1989), with parameters $\beta = 0.1$, $F = 0.1089$, $\omega = 0.85$. In the upper figure, the dark regions correspond to the basin of attraction in the vicinity of the well. The lower figure is a detail, highlighting the location of the resonant two-band chaotic attractor.

2. Most of the existing knowledge is too detailed to be of any engineering relevance. Identification of the destroyer saddle and the analysis of its complex invariant manifold structures can require care and effort even in the clean environment of a computer simulation. Very few physical engineering applications would possess the precision of control and lack of noise necessary to make such detailed analysis sensible.

3. Another problem may be that such events, being global bifurcations, are generally inaccessible to perturbative techniques. This is only a problem to those analysts who prefer such approaches. Szemplinska-Stupnicka (1988) has applied harmonic balance techniques to successfully approximate the occurrence of the nearby local bifurcation, namely the first flip of the resonant attractor. However, global perturbative methods similar to Melnikov's method, but aimed directly at the crisis itself, are less forthcoming since the orbits involved in such escape events typically traverse many energy levels of the underlying hamiltonian system.

It is in regard to these problems that this paper will propose a very general criterion associated with chaotic escape. This criterion will not depend on knowledge of the (as-yet not fully understood) fine details of the location and nature of the destroyer and its manifolds. The criterion will not be some algebraic expression but will be based on a simple observation of the topological configuration of certain manifolds. This configuration may be readily constructed and followed numerically

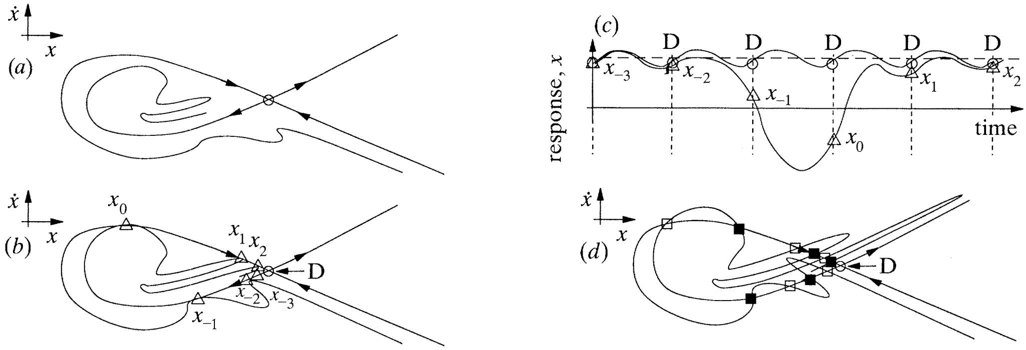

Figure 4. (*a*), (*b*), (*d*) are schematic illustrations of the invariant manifolds of the hilltop saddle D plotted on the Poincaré plane $(x, \dot{x})$ as parameters are varied through the initial homoclinic tangency (*b*). (*c*) shows the corresponding time-histories of the hilltop saddle cycle and the structurally unstable homoclinic orbit created at the tangency of (*b*). The triangular, circular and square markers denote Poincaré points of particular orbits.

using suitable algorithms (see, for example, Alexander 1989; Kawakami 1981). As its basis will be topological rather than being a perturbation of a nearby linear or hamiltonian system, the criterion will moreover not be restricted to the algebraically simple low-order polynomial functions that are necessary for perturbation theory, yet which are rarely encountered in engineering.

4. Homoclinic tangencies

In a dissipative system at small forcing amplitudes, the hilltop saddle cycle possesses no homoclinic orbit (see figure 4*a*). As the forcing amplitude is gradually increased the first homoclinic orbit is created at an event referred to as a homoclinic tangency (figure 4*b*). The stable and unstable manifolds of the hilltop saddle cycle touch at a (countably) infinite number of points, each such point being a Poincaré point of a single homoclinic orbit (see figure 4*c*) that starts arbitrarily close to the hilltop saddle, undergoes an excursion across the well and then reconverges back to the hilltop saddle. This orbit is structurally unstable, in that perturbations will typically either remove all homoclinicity or create two structurally stable (transverse) homoclinic orbits (figure 4*d*). This initial tangency is the one detected by Melnikov's perturbation method, which at low damping and forcing values, accurately predicts the parameter location of this event (Thompson 1989). However, this tangency alone has little to do with the chaotic escape of the resonant attractor or any other behaviour of the primary harmonic motions. At low damping values and forcing frequencies greater than resonance the 'chaos' associated with passage through this tangency consists of Smale horseshoes and associated cascades of subharmonic saddle-node and flip bifurcations which are typically confined to the extreme outer regions of the global basin, at energies typical of the hilltop saddle rather than the period one attractors. Any system following the period one attractor paths is energetically distanced from these regions and typically retains a simple period one motion throughout the tangency. At forcing frequencies above resonance, it is seen in figure 2*e* how the first flip (arc C) of the resonant attractor and the chaotic escape arc E are both typically well-distanced from the initial tangency (arc M). The primary chaotic escape event is thus clearly not simply a feature of passage through the initial homoclinic tangency.

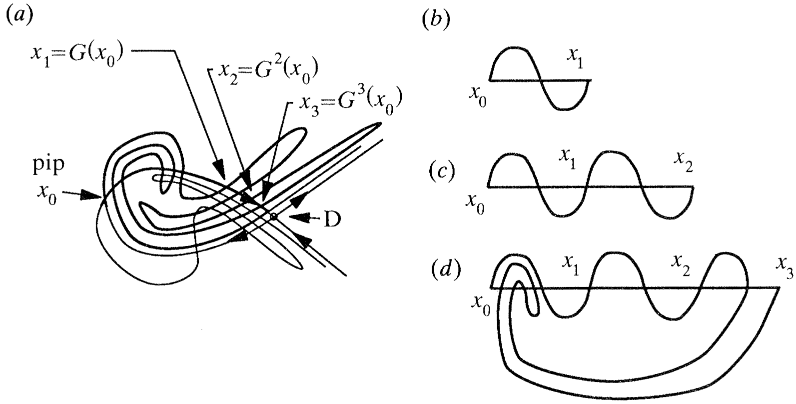

Figure 5. Examples of the Birkhoff signatures of a schematic homoclinic tangle (*a*). (*b*), (*c*), (*d*) show the period one, two and three signatures respectively. The period one and two signatures have yet to undergo their first changes, whereas the period three signature has already undergone at least two changes since the initial homoclinic tangency.

It will be shown, however, that there is another important homoclinic tangency in the manifolds of the hilltop saddle to which the chaotic escape event is more closely related.

5. Birkhoff signatures

As the control parameters are varied beyond the initial tangency, the stable and unstable manifolds of the hilltop saddle form a homoclinic tangle. Both curves become extremely convoluted and the global basin boundary (which is closely associated with the stable manifold) possesses a fractal structure. Under subsequent parameter ramping, the manifolds undergo a complex evolution involving infinite sequences of inner homoclinic tangencies with associated subharmonic cascades. This activity, initially in the outer regions of the global basin, progressively intrudes into the inner regions, a process referred to as global basin erosion by incursion of a fractal basin boundary.

In McRobie & Thompson (1991) it is described how the trellises formed by the homoclinically tangled invariant manifolds may be addressed by lobe dynamic techniques to provide a general understanding of the dynamical processes at work in such tangles. The structurally unstable homoclinic orbit created at the initial tangency degenerates into two structurally stable homoclinic orbits referred to as primary intersection points (or *pips*). A pip has the property that the stable and unstable manifold segments connecting the pip to the saddle do not intersect, other than at the pip and the saddle. Throughout the subsequent complex evolution of the manifolds the pips retain this property, and this simple invariance may be used to dissect the convoluted tangle into more manageable regions, referred to as *lobes*, whose fate under successive iterations of the map may be easily understood. The reader is referred to McRobie & Thompson (1991) for further details. In words, we select one of the two structurally stable homoclinic orbits emanating from the initial tangency, and select any of its Poincaré points from the infinite number present. The invariant manifold segments that connect this chosen pip to its nth image circumscribe and define the lobes, and the topological configuration of the lobes defines the period n Birkhoff signature. Some illustratory signatures are given in figure 5.

The period one Birkhoff signature is thus defined by the configuration of the lobes

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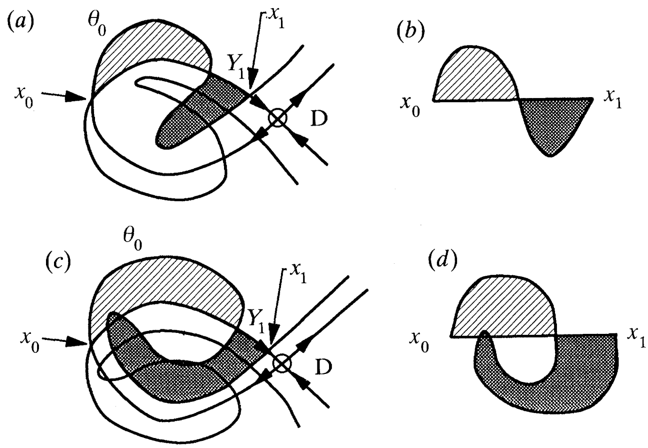

Figure 6. Schematic description of the first change of the period one Birkhoff signature. (*a*) shows homoclinically tangled invariant manifold structures and the associated period one Birkhoff signature (*b*) which has not yet undergone its first change. (*c*) shows the manifold structures after parameters have been varied to take the tangle marginally beyond its first period one Birkhoff signature change. This occurs when the lobe Y_1 first intersects the lobe Θ_0, this intersection creating a new homoclinic orbit at the tangency.

circumscribed by the invariant manifold segments that connect the pip to its image. On first passing through the initial homoclinic tangency, the period one signature necessarily has the form shown in figure 6*a*, with the two lobes, labelled Θ_0 and Y_1, necessarily disjoint. The first change in this signature occurs when the two lobes first intersect (see figure 6*b*), this corresponding to the passage through a particular inner homoclinic tangency.

The main result reported in this paper is that this first change in the period one Birkhoff signature is observed to occur close to (and shortly after) the chaotic escape of the primary resonant attractor. Some associated observations on the significance of such homoclinic tangencies (referred to as 'bifurcations of doubly asymptotic motions') have been previously highlighted in the work of Kawakami and co-workers (see, for example, Kawakami 1981) in investigations of nonlinear electronic engineering oscillators.

6. Numerical evidence

Two simple nonlinear systems having single-sided potential wells of the general form of figure 1 are considered; the 'generic' escape equation introduced by Thompson (1989):

(*a*) $$\ddot{x}+\beta\dot{x}+x-x^2 = F\sin\,(\omega t)$$

and a specific equation that arose in the study of a particular structural vibration problem involving both geometric and material stiffness nonlinearities:

(*b*) $$m\ddot{x}+c\dot{x}+K(x)\,x = F_s+F_d\cos\,(\omega_f t+\phi_0),$$

where $$K = \begin{cases} K_0, & |x| \leqslant x_{cr} \\ K_0\exp\,(-\alpha(|x|-x_{cr})), & |x| > x_{cr} \end{cases}.$$

Bifurcation diagrams for the two systems are given in figure 7*a*, *b*. Not only are the potential wells of the two systems mathematically dissimilar in formulation, but the

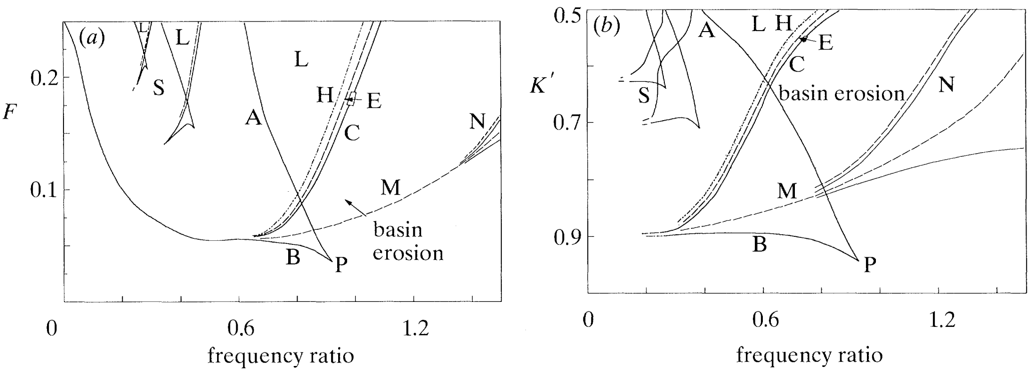

Figure 7. Bifurcation diagrams for the two equations of §6. (*a*) is for $\beta = 0.1$, (*b*) is for $m = 1$, $c = 0.02$, $K_0 = 1$, $x_{cr} = 1$, $F_s = 0.95$, $F_d/F_s = 16$. Both diagrams exhibit: the cusp P where the saddle-node bifurcation arcs A and B meet; the arc C of the first period-doubling bifurcation above the initial homoclinic tangency arc M; slightly above the arc C is the chaotic escape arc E where the chaotic attractor formed by the cascade starting at C loses stability at a crisis. In both systems the arc H, the locus of the first change in the period one Birkhoff signature, is seen to lie near to and slightly above the crisis are E. The approximate location of some of the superharmonic and subharmonic resonances are shown at S and N respectively, where similar cascade/crisis/signature change phenomena are observed.

parameter sets underlying the two bifurcation diagrams are also substantially dissimilar. There is a factor of five difference in the damping ratios, and the ordinate in the bifurcation diagram of the generic system is taken as the applied forcing amplitude, whereas for the specific system the ordinate is taken as the stiffness decay term $K' = e^{-\alpha}$. Both ordinates if suitably non-dimensionalized would represent increasing levels of forcing amplitude in a potential well of fixed depth. Historically the two diagrams represent results from different research directions. The choice of these parameter sets is thus not prearranged to exhibit similar bifurcational phenomena. The observation that the bifurcational structures of the two systems bear strong similarities thus provides evidence to support the assertion that generic patterns of bifurcation are being observed. This is further to the results of Stewart *et al.* (1991) and those of Lansbury & Thompson (1990) who observed such similarities between cubic and quartic potential functions. It similarly follows that if the association between the signature change and the final boundary crisis can be established in both systems, it would likewise provide strong evidence that such is a robust and general mechanism for the escape of a resonant attractor.

The bifurcation diagrams were constructed by a variety of methods of numerical exploration, including path-following, local bifurcation-following and cell-to-cell mapping techniques (Hsu 1987). Preliminary estimates of the location of Birkhoff signature changes were obtained by inspection of unstable manifolds superimposed on cell maps. These manifolds were calculated by the standard technique of locating the hilltop saddle cycle with a simple Newton–Raphson algorithm and extrapolating unstable manifolds by forward iteration of a ladder of starts on a fundamental domain on the outgoing eigenvectors close to the saddle. In cases where the cell mapping located no attractor but the hilltop saddle exists, the stable manifold was similarly computed by backward iteration of a fundamental domain on the ingoing eigenvectors. Detailed location of Birkhoff signature changes was achieved using a dedicated program, similar to the general homoclinic tangency following code of

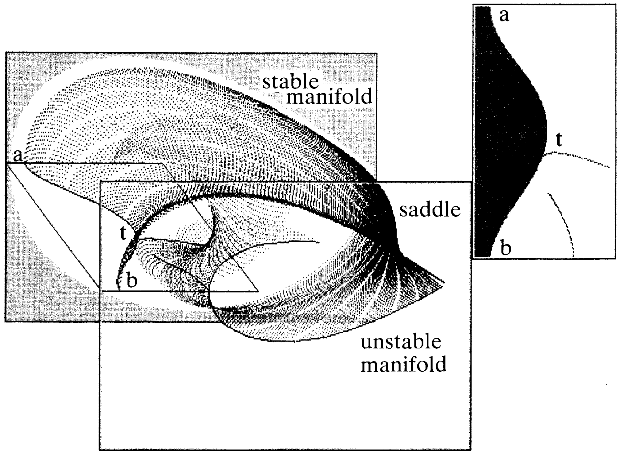

Figure 8. Computer realisation of the invariant manifolds used in the detection of Birkhoff signature change. The stable manifold is extrapolated to the curve atb (corresponding to C_s). Construction of the unstable manifold has located an orbit that crosses to the right of atb, but subsequently reaches atb less than one period later. Frequency is 0.85, force is 0.124.

Alexander (1989), but implemented in such a manner as to ensure that the particular tangency being followed is indeed that of the desired Birkhoff signature change. Broadly, the algorithm extrapolates the stable and unstable manifolds away from the saddle to define curves C_s and C_u where they intersect a surface of known velocity (either the zero velocity plane, or preferably the velocity of the hilltop saddle at that phase). Parameters are then adjusted until an orbit in C_u can first land to the right of C_s but subsequently reach C_s within one period of the applied forcing. Some ancillary computer graphics generated in the course of implementation of the algorithm are shown in figure 8, where intersection points and tangencies are shown using the full three-dimensional space $(x, \dot{x}, \phi)$, rather than restricting attention to a single preselected Poincaré plane. Further details of the construction will appear in McRobie & Thompson (1992*a*).

The parameter locations of the initial period 1 signature change computed by this algorithm are plotted on the corresponding bifurcation diagrams in figure 7*a*, *b*. In both systems the association between the chaotic escape arc of the resonant attractor and the signature change is evident. In both systems under increasing 'force', the chaotic escape event precedes the signature change.

7. Engineering perspective

If the applied forcing is such that the hilltop saddle cycle is substantially localized around the crest of the potential maximum then the initial homoclinic tangency of the manifolds of that saddle corresponds to the first occurrence of a trajectory which, starting arbitrarily close to the hilltop saddle cycle, falls away along the outgoing eigenvector, traverses the potential well once and then converges back to the hilltop saddle along the ingoing eigenvector. The infinite set of homoclinic tangency points where the stable and unstable manifolds intersect all correspond to this single orbit (figure 4*c*).

The single orbit that corresponds to the initial change of the period one Birkhoff signature is very similar (figure 9*a*), but having reached the far side of the well,

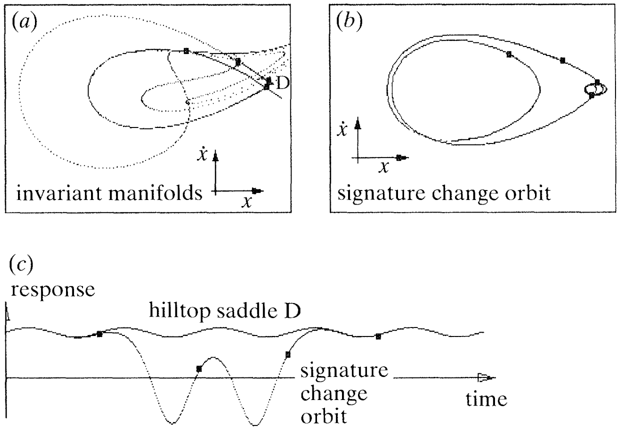

Figure 9. The nature of the homoclinic orbit created at the first period one signature change. The black squares correspond to Poincaré points of this orbit, located in (*a*) with respect to the invariant manifolds on that Poincaré plane, in (*b*) with respect to the projection of the full orbit onto that plane, and in (*c*) with respect to a time history of the response. All curves are computed for the equation (*a*) of §6 with $\omega = 0.85$, $F = 0.124$ and $\beta = 0.1$, and correspond to the invariant manifolds of figure 8.

instead of returning directly to the hilltop, the orbit makes an additional circuit around the well (figure 9*b*). It should be noted that although the time taken to cross from the saddle to the far side of the well is infinite, the additional circuit around the well must be completed within one period of the forcing.

One simple implication that may be drawn, for example, is that for a given amplitude of forcing, if a dynamical system is at a frequency substantially higher than resonance and at some lower frequency there exists a single orbit of the form of figure 9*c* that can fall from the hilltop, complete the additional circuit within one period and return to the hilltop, then as the frequency is decreased that system is liable to escape. Such an inference is to some extent not rigorous, but the underlying idea that global information about system stability is provided by knowledge of the existence of a single orbit remains. From simple numerical procedures involving only the location of the hilltop saddle and construction of sufficient length of invariant manifold to establish the lobe configuration of the period one Birkhoff signature, it is thus contended that significant information about the dynamic stability of evolving systems can be obtained, without reference to the fine details of possibly chaotic attractor structures and destroyer collisions.

8. Dynamical systems theory perspective

From the Smale–Birkhoff homoclinic theorem, the creation of transverse homoclinic orbits by the passage through homoclinic tangencies is associated with horseshoe formation, which in turn is associated with cascades of periodic bifurcations to populate the invariant sets of the horseshoes. Much theoretical work has been undertaken to establish the nature of the associated subharmonic activity in such processes (see, for example, Gavrilov & Silnikov 1983; Yorke & Alligood 1983; Newhouse 1980; Patterson & Robinson 1985). In the light of such work, the observation reported here can be to some extent understood by observing that the passage through the period one Birkhoff signature creates a 'period one horseshoe' (see figure 10) which possesses two period one saddles. One of these is directly

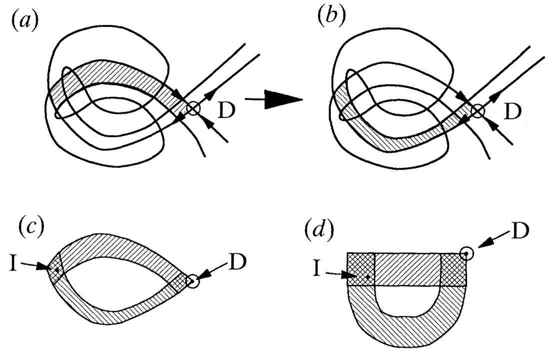

Figure 10. Schematic illustration of the formation of a simple horseshoe at the first change of the period one signature. The shaded region in (*a*) maps under one forcing cycle to the shaded region of (*b*), the intersection of these two regions is sketched in (*c*), this being topologically equivalent to the Smale horseshoe of (*d*). The period one inversely unstable saddle I whose existence and location is readily anticipated by a simple horseshoe analysis of (*d*) corresponds to the saddle originating at the first period-doubling of the resonant attractor C, figures 2 and 7).

unstable, and corresponds to the hilltop saddle, the other is inversely unstable and is that which originates from the period-doubling of the resonant attractor.

Analysis of this association between horseshoes and tangencies may be taken to arbitrary detail. In McRobie & Thompson (1992*b*) it is described how a general overview of the global dynamics involving most major subharmonic and superharmonic bifurcational activity may be obtained by simply looking at the sequences of higher order Birkhoff signature changes that must occur as parameters are varied. One particular observation is that the initial change in the period n Birkhoff signature is associated with the chaotic escape of a period n solution which may be identified as the period n simple Newhouse orbit. In light of this and the main result of this paper, it is interesting to identify the primary resonant attractor of such simple nonlinear oscillators as the period one simple Newhouse orbit. In a forthcoming paper this identification will be strengthened by the use of symbolic dynamics to compute rotation matrices for periodic and homoclinic orbits. In particular a precedence relation between the period-doubling cascade of the period n simple Newhouse orbit and the initial change in the period n Birkhoff signature change will be derived, which in the case of $n = 1$, is directly applicable here.

9. Summary

The familiar chaotic escape of the resonant period one attractor in a wide class of simple nonlinear oscillators is related to the initial change in the period one Birkhoff signature of the invariant manifolds of the hilltop saddle. This signature change corresponds to the existence of a single orbit. The existence or otherwise of such an orbit is readily discerned numerically and may thus be of some utility in investigating the failure of engineering dynamical systems. Since the orbit is a structurally unstable homoclinic curve, the chaotic escape of the resonant attractor may be better understood when seen as a topological feature of the global dynamics. Such an approach does not depend upon the system being close to a linear or a hamiltonian system, not does it require the algebraically simple potential and forcing functions that are typically necessary for manipulation by standard perturbation techniques.

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PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES